Boundary conditions from boundary terms, Noether charges and the trace K lagrangian in general relativity.

Josep M. Pons[a]

[a]*Departament d'Estructura i Constituents de la Matèria*
Universitat de Barcelona
Av. Diagonal 647
08028 Barcelona
Catalonia, Spain

e-mail: pons@ecm.ub.es

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Abstract

We present the Lagrangian whose corresponding action is the trace K action for General Relativity. Although this Lagrangian is second order in the derivatives, it has no second order time derivatives and its behavior at space infinity in the asymptotically flat case is identical to other alternative Lagrangians for General Relativity, like the gamma-gamma Lagrangian used by Einstein. We develop some elements of the variational principle for field theories with boundaries, and apply them to second order Lagrangians, where we stablish the conditions —proposition 1— for the conservation of the Noether charges. From this general approach a pre-symplectic form is naturally obtained that features two terms, one from the bulk and another from the boundary. When applied to the trace K Lagrangian, we recover a pre-symplectic form first introduced using a different approach. We prove that all diffeomorphisms satisfying certain restrictions at the boundary —that keep room for a realization of the Poincaré group— will yield Noether conserved charges. In particular, the computation of the total energy gives, in the asymptotically flat case, the ADM result.

1 Introduction

Either way, boundary conditions *from* boundary terms or boundary terms *for* boundary conditions; boundary terms for the action and boundary conditions for the field configurations go together. Whereas from a purely logical point of view the boundary conditions emerge as a consequence of the boundary terms, from the technical side the situation is often the opposite: one must look for the boundary terms that are necessary to implement the desired —or acceptable— boundary conditions.

This discussion applies directly to General Relativity (GR). It is well known that the boundary conditions required for the correct application of the variational principle in GR depend upon the boundary terms exhibited by the action. Since divergence terms do not alter Einstein equations of motion, there is some freedom to write down Lagrangians for GR that differ in

some divergence terms from the original Einstein-Hilbert proposal. These divergence terms lead to different boundary terms for the action. To be specific, consider the original Einstein-Hilbert action (integration is in a 4-volume ${}^4V$ of spacetime)

$$S_{EH} = \int_{{}^4V} d^4x \, \sqrt{|g|} R \, . \tag{1}$$

In this case, the boundary conditions for the application of the variational principle involve the fixation of some derivatives of the metric at the boundary. Instead, subtraction of a divergence from (1) allows to write the "gamma-gamma" action, first used by Einstein,

$$S_{\mathbf{\Gamma}} = \int_{{}^4V} d^4x \, \sqrt{|g|} g^{\mu\nu} (\mathbf{\Gamma}^\rho_{\mu\sigma} \mathbf{\Gamma}^\sigma_{\rho\nu} - \mathbf{\Gamma}^\rho_{\mu\nu} \mathbf{\Gamma}^\sigma_{\rho\sigma}) \, , \tag{2}$$

which is first order in the spacetime derivatives and that has the possible advantage that the variational principle only requires to fix the metric at the boundary.

In field theory, the association –by means of the Noether theorem– of symmetries at the level of the variational principle with conserved currents makes the role of boundary terms (leading to boundary conditions for the fields), either in the Lagrangian or the Hamiltonian formulation, essential for the conservation of charges, as shown in the pioneering work of [1] for GR (see also recent work in [2, 3, 4]). The role of boundary terms has also been stressed in recent years by the introduction of the concept of quasilocal charges in GR (see [5] for general references, see also [6, 7, 8]).

Many actions, all differing from Einstein-Hilbert's in boundary terms, have been used in the literature. With no aim of completeness, let us mention contributions in [9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20] and many references therein. Among them, and using the terminology of [8], the "trace K" action [11, 13, 8] offers some singular features, because of its geometric character, that deserve full interest. This action is roughly written as

$$S_K = \int_{{}^4V} d^4x \, \sqrt{|g|} R - 2 \int_{\partial {}^4V} d^3x \, \sqrt{|\gamma|} K \, , \tag{3}$$

where $\partial {}^4V$ is the boundary of the 4-volume ${}^4V$, γ is the determinant of the 3-metric induced on the boundary and K is its extrinsic curvature. When the boundary is not smooth, and discontinuities arise in the vector field orthonormal to the boundary, the second term in (3) must be understood [15] as including delta-like contributions from the "joints" between the smooth elements of the boundary. The detailed form of S_K is given in section 3.

The purpose of the present paper is to analyse the Lagrangian leading to the action (3) and to provide with the theoretical framework for actions of this type. In section 2 we introduce some notation and useful formulas. The Lagrangian for the trace K action is obtained in section 3. In section 4 we develop the formal theory for the variational principle and the Noether symmetries for second order Lagrangians in field theories with boundaries. This theory is applied in section 5 to the trace K Lagrangian and we compute the total energy in the particular case of asymptotically flat spaces. Conclusions are presented in section 6 and the appendix is devoted to prove some expressions used in the text.

2 Some notation and formulas

Here we introduce some notation and formulas to be used in the next sections. We consider that our spacetime coordinates correspond to a standard $3+1$ decomposition, including in particular that $g^{00} < 0$ and $g^{ii} > 0$, that is, no surface $x^\mu = constant$ is tangent to the light cone. Working with this type of coordinates means no restriction, at least infinitesimally, on the gauge freedom, because any infinitesimal diffeomorphism preserves these conditions for the metric components. The determinant of the 4-metric $g_{\mu\nu}$ will be denoted by g.

Given the 3-surface $x^\mu = constant$, its orthonormal vector $\mathbf{n}(\mu)$ is defined by the relation

$$n^\mu(\mu) n^\nu(\mu) = \xi(\mu) g^{\mu\nu} \ ,$$

and by requiring that it is pointing towards increasing values of the coordinate x^μ. The coefficient $\xi(\mu)$ is just a sign, $\xi(\mu) = \eta_{\mu\mu}$, where $\eta_{\mu\nu}$ is the Minkowski metric with positive signature $(-,+,+,+)$. Then,

$$n^\nu(\mu) = \xi(\mu) \frac{g^{\mu\nu}}{\sqrt{|g^{\mu\mu}|}}$$

Let us point out that $\mathbf{n}(\mu)$ is not a true vector field for it fails to transform as such under the diffeomorphisms that do not preserve the foliation defined by the 3-surfaces $x^\mu = constant$. We compute in the appendix the deviation from the vector behavior of the transformation of $\mathbf{n}(\mu)$ under diffeomorphisms.

The 3-metric induced at the 3-surface $x^\mu = constant$ is g_{ab}, with $a, b = 0, 1, 2, 3$ *except* μ. Its inverse matrix is given by $\gamma^{ab}(\mu)$; the components of this matrix are the non identically vanishing components of

$$\gamma^{\rho\sigma}(\mu) := g^{\rho\sigma} - \frac{g^{\rho\mu} g^{\mu\sigma}}{g^{\mu\mu}} = g^{\rho\sigma} - \xi(\mu) n^\rho(\mu) n^\sigma(\mu). \tag{4}$$

The determinant $\det g_{ab}$ (for $a, b = 0, 1, 2, 3$ *except* μ) will be denoted $\gamma(\mu)$. There is the relationship

$$g g^{\mu\mu} = \gamma(\mu).$$

Consider, for $\nu \neq \mu$, the 2-surface $x^\mu = constant, \ x^\nu = constant$. The induced metric on it is g_{AB}, with $A, B = 0, 1, 2, 3$ *except* μ, ν. Its inverse metric will be written as $\gamma^{AB}(\mu\nu)$, and the determinant $\det g_{AB} =: \gamma(\mu\nu)$. One can show:

$$g(g^{\mu\mu} g^{\nu\nu} - (g^{\mu\nu})^2) = g g^{\mu\mu} \gamma^{\nu\nu}(\mu) = \gamma(\mu\nu).$$

Note also that

$$\gamma^{\nu\nu}(\mu) = g^{\nu\nu}(1 - \frac{(g^{\mu\nu})^2}{g^{\mu\mu} g^{\nu\nu}}) = g^{\nu\nu}(1 - \xi(\mu)\xi(\nu) q^2(\mu\nu)), \tag{5}$$

where we have defined the scalar products

$$q(\mu\nu) = \frac{g^{\mu\nu}}{\sqrt{\xi(\mu)\xi(\nu) g^{\mu\mu} g^{\nu\nu}}} = \xi(\mu)\xi(\nu) \mathbf{n}(\mu) \cdot \mathbf{n}(\nu) \ .$$

The extrinsic curvature for the surface $x^\mu = constant$ is given by

$$K_{ab}(\mu) := -\frac{1}{\sqrt{|g^{\mu\mu}|}}\mathbf{\Gamma}^\mu_{ab} \ ,$$

which is equivalent to

$$K_{ab}(\mu) = \frac{1}{2}\xi(\mu)\mathcal{L}_{\mathbf{n}(\mu)} g_{ab} \ ,$$

where $\mathcal{L}_{\mathbf{n}(\mu)}$ is the Lie derivative under $\mathbf{n}(\mu)$.

The trace of the extrinsic curvature $K(\mu) := \gamma^{ab}(\mu)K_{ab}(\mu)$ may be written as

$$K(\mu) = \xi(\mu)\mathcal{L}_{\mathbf{n}(\mu)}(ln\sqrt{|g|}) = \xi(\mu)n^\nu_{;\nu}(\mu) \ ,$$

where $n^\nu_{;\nu}(\mu)$ is the covariant derivative of $\mathbf{n}(\mu)$, $\nabla_\nu n^\nu(\mu)$, with the Riemannian connexion, as if $\mathbf{n}(\mu)$ were a true vector.

3 The Lagrangian for the trace K action

In this section we use techniques introduced in [21] (see also[22, 23]). Let us first take a look on the boundary conditions imposed by the variational principle for the Einstein-Hilbert Lagrangian. A general variation for $\mathcal{L}_{EH}$ gives

$$\delta\mathcal{L}_{EH} = -G^{\mu\nu}\delta g_{\mu\nu} + \sqrt{|g|}g^{\mu\nu}\delta R_{\mu\nu}, \tag{6}$$

where $R_{\mu\nu}$ stands for the Ricci tensor and

$$G^{\mu\nu} := \sqrt{|g|}(R^{\mu\nu} - \frac{1}{2}g^{\mu\nu}R),$$

with R the scalar curvature.

As it is well known, the last term in (6) is a divergence. It can be written as

$$\sqrt{|g|}g^{\mu\nu}\delta R_{\mu\nu} = \partial_\mu(\tilde{g}^{\rho\nu\mu}_\sigma \delta\mathbf{\Gamma}^\sigma_{\rho\nu}) \ , \tag{7}$$

where $\tilde{g}^{\rho\nu\mu}_\sigma := \sqrt{|g|}(g^{\rho\nu}\delta^\mu_\sigma - g^{\mu(\nu}\delta^{\rho)}_\sigma)$. The next step is to write [21] $\tilde{g}^{\rho\nu\mu}_\sigma \delta\mathbf{\Gamma}^\sigma_{\rho\nu}$ as (we continue with the convention of indices $a, b = 0, 1, 2, 3$ *except* μ)

$$\tilde{g}^{\rho\nu\mu}_\sigma \delta\mathbf{\Gamma}^\sigma_{\rho\nu} = -g_{ab}\delta P^{ab}(\mu) + \partial_\nu(\sqrt{|g|}g^{\mu\mu}\delta(\frac{g^{\mu\nu}}{g^{\mu\mu}})) \ , \tag{8}$$

where $P_{ab}(\mu)$ (for a given μ, notice that we raise and lower indices with $g_{ab}, \gamma^{ab}(\mu)$) is a generalisation to any μ of the ADM [24] momenta for $\mu = 0$:

$$P_{ab}(\mu) = \sqrt{|\gamma(\mu)|}(g_{ab}K(\mu) - K_{ab}(\mu)).$$

Therefore, the divergence term in (7) is

$$\begin{aligned}\partial_\mu(\tilde{g}^{\rho\nu\mu}_\sigma \delta\mathbf{\Gamma}^\sigma_{\rho\nu}) &= -\partial_\mu(g_{ab}(\mu)\delta P^{ab}(\mu)) + \partial_\mu\partial_\nu\left(\sqrt{|g|}g^{\mu\mu}\delta(\frac{g^{\mu\nu}}{g^{\mu\mu}})\right) \\ &= -\partial_\mu(g_{ab}(\mu)\delta P^{ab}(\mu)) + \frac{1}{2}\partial_\mu\partial_\nu(\sqrt{|g|}\left(g^{\mu\mu}\delta(\frac{g^{\mu\nu}}{g^{\mu\mu}}) + g^{\nu\nu}\delta(\frac{g^{\mu\nu}}{g^{\nu\nu}})\right)).\end{aligned} \tag{9}$$

Finally, the term acted upon by the derivatives $\partial_\mu \partial_\nu$ has the equivalent expression:

$$\begin{aligned}
\frac{1}{2}\sqrt{|g|}\left(g^{\mu\mu}\delta(\frac{g^{\mu\nu}}{g^{\mu\mu}})+g^{\nu\nu}\delta(\frac{g^{\mu\nu}}{g^{\nu\nu}})\right) &= \left(\sqrt{|g|}\sqrt{\xi(\mu)\xi(\nu)g^{\mu\mu}g^{\nu\nu}}\right)\delta\left(\frac{g^{\mu\nu}}{\sqrt{\xi(\mu)\xi(\nu)g^{\mu\mu}g^{\nu\nu}}}\right) \\
&= \left(\sqrt{|g|}\sqrt{\xi(\mu)\xi(\nu)g^{\mu\mu}g^{\nu\nu}}\right)\delta q(\mu\nu) \\
&= \sqrt{|\gamma(\mu\nu)|}\delta\alpha(\mu\nu) \ , \qquad (10)
\end{aligned}$$

where the angles $\alpha(\mu\nu)$ are defined as

$$\alpha(0i) = \text{arcsinh}(q(0i)) \ , \quad \alpha(ij) = \arcsin(q(ij)) \ ,$$

and we have used

$$\delta\alpha(\mu\nu) = \frac{\delta q(\mu\nu)}{\sqrt{(1-\xi(\mu)\xi(\nu)q^2(\mu\nu))}} \ .$$

Therefore

$$\delta S_{EH} = \int_{^4V} d^4x\ \delta\mathcal{L}_{EH} = \int_{^4V} d^4x\ \left(-G^{\mu\nu}\delta g_{\mu\nu} - \partial_\mu(g_{ab}(\mu)\delta P^{ab}(\mu)) + \partial_\mu\partial_\nu(\sqrt{|\gamma(\mu\nu)|}\delta\alpha(\mu\nu))\right) \qquad (11)$$

Consider the volume ^4V being a 4-cube, having as boundary elements 3-surfaces defined by the constancy of one of the coordinates x^μ. Then the second term in the integrand will give a contribution from the faces and the third a contribution from the joints, these joints being defined by the constancy of two coordinates. In order for S_{EH} to be a differentiable functional [25], and its variation to consequently lead to Einstein equations, we need to control the fields and the variations at the boundary in such a way that

$$\delta S_{EH} = 0 \Longleftrightarrow G^{\mu\nu} = 0.$$

As we have already said, this control at the boundary involves some derivatives of the metric components. Now we have the specifics: the variation of $P^{ab}(\mu)$ must vanish at the faces $x^\mu = constant$, and the variation of $\alpha(\mu\nu)$ must vanish at the joints between the faces $x^\mu = constant$ and $x^\nu = constant$. The physical meaning of this restrictions on the variations is unclear.

Instead, a simple addition of a divergence term to $\mathcal{L}_{EH}$ will give a more reasonable control of the variations at the boundary. Taking into account that $g_{ab}(\mu)P^{ab}(\mu) = 2\sqrt{|\gamma(\mu)|}K(\mu)$, define the new Lagrangian (K is for the extrinsic curvature)

$$\mathcal{L}_K := \mathcal{L}_{EH} + 2\partial_\mu(\sqrt{|\gamma(\mu)|}K(\mu)) - \partial_\mu\partial_\nu(\sqrt{|\gamma(\mu\nu)|}\alpha(\mu\nu)) \ . \qquad (12)$$

The action (3) is, by definition, $S_K := \int_{^4V} d^4x\ \mathcal{L}_K$. Then,

$$\delta S_K = \int_{^4V} d^4x\ \delta\mathcal{L}_K = \int_{^4V} d^4x\ \left(-G^{\mu\nu}\delta g_{\mu\nu} + \partial_\mu(P^{ab}(\mu)\delta g_{ab}(\mu)) - \partial_\mu\partial_\nu(\alpha(\mu\nu)\delta\sqrt{|\gamma(\mu\nu)|})\right) \qquad (13)$$

So for S_K to be a differentiable functional we must require, besides the customary vanishing of the variations of the 3-metric induced on the initial and final equal-time 3-surfaces, the

supplementary condition of the vanishing of the variations of the 3-metric induced on any spatial 3-face of the boundary. These requirements already guarantee the fixation of the variations of the determinant $\gamma(\mu\nu)$ at the joints. In fact, to be more precise, the vanishing of the variations of the 3-metric is only required for finite boundaries; in the case of asymptotically flat spaces, where we let the spatial elements of the boundary go to the space infinity ($r \to \infty$), we must control the $r \to \infty$ behavior of the fields and its allowed variations so that there is no contribution to δS_K. More on this later.

Notice the difference with the boundary conditions that one finds in the case of the gamma-gamma Lagrangian, where the vanishing of the variations of the 4-metric is required.

Expression (12) is the Lagrangian for the action (3). We notice that the contributions from the joints are all included. The role of these contributions has been explained in [15]. Early computations can be found in [22, 23].

Two features of (12) are worth being mentioned immediately. First, as it happens generally with the proposals to modify $\mathcal{L}_{EH}$ through divergence terms, $\mathcal{L}_K$ is not a truly scalar density. This fact does not opposes to its physical applicability, as it is argued in a parallel context in [26] for the gamma-gamma Lagrangian. $\mathcal{L}_K$ gives indeed the trace K action, with all its geometric meaning, but for coordinates adapted to the boundary –or viceversa: for boundaries adapted to the coordinates–, in such a way that the elements of the boundary correspond to the constancy of the value of some coordinate. A typical boundary may be the 4-cube, already used, but a cylinder whose top and bottom faces are equal-time 3-surfaces, and whose lateral face is defined by the constancy of a single radial coordinate, is a well adapted boundary as well. In the cylinder case the (ij) (i, j is for space indices) contributions from the last term in (13) disappear in S_K because "the boundary of a boundary is zero".

The second observation is that, unlike the "gamma-gamma" Lagrangian (2), $\mathcal{L}_K$ is not a first order Lagrangian. This is a point not sufficiently recognised in the literature [10, 13, 27, 28]. It is proved in the appendix that the second order contributions to $\mathcal{L}_K$ are as follows:

$$\mathcal{L}_K = (\text{quadratic terms in the first derivatives of the metric}) - \alpha(\mu\nu)\partial_\mu\partial_\nu\sqrt{|\gamma(\mu\nu)|}, \quad (14)$$

but note a key difference with the Einstein-Hilbert Lagrangian: $\mathcal{L}_K$ has no second order time derivatives (the sum μ, ν in (14) only contributes for $\mu \neq \nu$). In this sense $\mathcal{L}_K$ has an intermediate place between $\mathcal{L}_{EH}$ (with second order time derivatives) and $\mathcal{L}_{\mathbf{\Gamma}}$ (with only first order spacetime derivatives). In the next section we will introduce some notation and results for theories with second order Lagrangians to focus later on Lagrangians of the type of $\mathcal{L}_K$. The advantage of not having second order time derivatives will become clear when we analyse the boundary conditions. Also, we will see in the asymptotically flat case that the long distance behavior of $\mathcal{L}_K$ improves crucially that of $\mathcal{L}_{EH}$.

4 Second order Lagrangians: variational principle for field theories with boundaries

Here we will consider a generic second order Lagrangian density function $\mathcal{L}$ with dependences:

$$\mathcal{L}(\phi, \phi_\mu, \phi_{\mu\nu})$$

ϕ denotes the whole set of fields (a new index could be introduced but it will be unnecessary). ϕ_μ stands for $\partial_\mu\phi$, and $\phi_{\mu\nu} := \partial_\mu\partial_\nu\phi$. If, as it happens with $\mathcal{L}_K$, only terms with $\mu \neq \nu$ appear in the second derivatives, some of the expressions simplify somewhat.

The Lagrangian functional is

$$L[\phi, \dot\phi, \ddot\phi] = \int_{^3V} d^3x \; \mathcal{L}(\phi, \phi_\mu, \phi_{\mu\nu}),$$

for some spatial 3-volume, and the action functional is

$$S[\phi] = \int_{t_0}^{t_1} dt \; L[\phi, \dot\phi, \ddot\phi] = \int_{^4V} d^4x \; \mathcal{L}(\phi, \phi_\mu, \phi_{\mu\nu}),$$

with $^4V = [t_0, t_1] \times {^3V}$. The functional differentiation of a general S is given [29] by

$$\delta S = \int d^4x \; \partial^{(m)} \left(\frac{\delta S}{\delta \phi^{(m)}} \delta\phi \right) \tag{15}$$

(m is a condensed notation for any number of partial derivatives with respect to any spacetime coordinate), that must be compared with

$$\delta S = \int d^4x \; \delta\mathcal{L} = \int d^4x \; \frac{\partial \mathcal{L}}{\partial \phi^{(m)}} \delta\phi^{(m)} \tag{16}$$

in order to define the functional derivatives $\frac{\delta S}{\delta \phi^{(m)}}$. We will ignore, in the way of computing variations with respect to the second order derivatives, the symmetry $\phi_{\mu\nu} = \phi_{\nu\mu}$. This means that we compute independently, for instance, $\frac{\delta S}{\delta\phi_{\mu\nu}}$ and $\frac{\delta S}{\delta\phi_{\nu\mu}}$. It is useful then to define the "complete" derivative —that takes into account the truly independent variables—,

$$\frac{\delta^c S}{\delta \phi_{\mu\nu}} = \frac{\delta^c S}{\delta \phi_{\nu\mu}} := \frac{\delta S}{\delta \phi_{\mu\nu}} + \frac{\delta S}{\delta \phi_{\nu\mu}},$$

for $\mu \neq \nu$, and

$$\frac{\delta^c S}{\delta \phi_{\mu\mu}} = \frac{\delta S}{\delta \phi_{\mu\mu}}$$

for the rest.

Comparing (15) and (16) gives,

$$\frac{\delta S}{\delta \phi} = \frac{\partial \mathcal{L}}{\partial \phi} - \partial_\mu\Big(\frac{\partial \mathcal{L}}{\partial \phi_\mu}\Big) + \partial_\nu\partial_\mu\Big(\frac{\partial \mathcal{L}}{\partial \phi_{\mu\nu}}\Big) \tag{17}$$

$$\frac{\delta S}{\delta \phi_\mu} = \frac{\partial \mathcal{L}}{\partial \phi_\mu} - \partial_\nu\Big(\frac{\partial \mathcal{L}}{\partial \phi_{\mu\nu}} + \frac{\partial \mathcal{L}}{\partial \phi_{\nu\mu}}\Big) \tag{18}$$

$$\frac{\delta S}{\delta \phi_{\mu\nu}} = \frac{\partial \mathcal{L}}{\partial \phi_{\mu\nu}} \; . \tag{19}$$

The first line displays the Euler-Lagrange derivatives, yielding the equations of motion when set to zero.

Analogous variations can be computed for $L[\phi, \dot\phi, \ddot\phi]$, taking into account that ϕ, $\dot\phi$ and $\ddot\phi$ are independent arguments for L. It is convenient to define

$$\frac{\partial \mathcal{L}}{\partial \dot\phi_i} := \frac{\partial \mathcal{L}}{\partial \phi_{0i}} + \frac{\partial \mathcal{L}}{\partial \phi_{i0}}, \quad \frac{\delta S}{\delta \dot\phi_i} := \frac{\delta^c S}{\delta \phi_{0i}}, \quad \frac{\delta L}{\delta \dot\phi_i} := \frac{\delta^c L}{\delta \phi_{0i}} \; .$$

For the sake of completeness we give the relationship between functional derivatives for S and L. They are

$$\frac{\delta S}{\delta \phi} = \frac{\delta L}{\delta \phi} - \partial_0(\frac{\delta L}{\delta \dot{\phi}}) + \partial_0\partial_0(\frac{\delta L}{\delta \ddot{\phi}}), \quad \frac{\delta S}{\delta \dot{\phi}} = \frac{\delta L}{\delta \dot{\phi}} - 2\partial_0(\frac{\delta L}{\delta \ddot{\phi}}), \quad \frac{\delta S}{\delta \phi_i} = \frac{\delta L}{\delta \phi_i} - \partial_0(\frac{\delta L}{\delta \dot{\phi}_i}),$$

and

$$\frac{\delta S}{\delta \dot{\phi}_i} = \frac{\delta L}{\delta \dot{\phi}_i}, \quad \frac{\delta S}{\delta \phi_{ij}} = \frac{\delta L}{\delta \phi_{ij}} \ .$$

Now we will explore the requirement of differentiability coming from the variational principle. Using (16)

$$\begin{aligned} \delta S &= \int_{^4V} d^4x \, \left(\frac{\delta S}{\delta \phi}\delta\phi + \partial_\mu(\frac{\delta S}{\delta \phi_\mu}\delta\phi) + \partial_\nu\partial_\mu(\frac{\delta S}{\delta \phi_{\mu\nu}}\delta\phi) \right) \\ &= \int_{^4V} d^4x \, \frac{\delta S}{\delta \phi}\delta\phi + \left[\int_{^3V} d^3x \, \frac{\delta S}{\delta \dot{\phi}}\delta\phi \right]_{t=t_0}^{t=t_1} + \int_{t_0}^{t_1} dt \int_{\partial^3 V} d\sigma_i \, \frac{\delta S}{\delta \phi_i}\delta\phi \\ &+ \left[\int_{\partial^3 V} d\sigma_i \, \frac{\delta S}{\delta \dot{\phi}_i}\delta\phi \right]_{t=t_0}^{t=t_1} + \int_{^4V} d^4x \, \partial_i\partial_j(\frac{\delta S}{\delta \phi_{ij}}\delta\phi) + \left[\int_{^3V} d^3x \, \partial_0(\frac{\delta S}{\delta \phi_{00}}\delta\phi) \right]_{t=t_0}^{t=t_1} , \end{aligned} \tag{20}$$

where $\partial^3 V$ is the –spatial– boundary of ${}^3V$ and $d\sigma_i$ are the two-forms induced at the boundary by the application of Stokes theorem.

The second term, the fourth term, and a piece within the sixth term, in the last equality, vanish because $\delta\phi|_{t_0} = \delta\phi|_{t_1} = 0$ as conditions imposed on the variations allowed by the variational principle. So in order for the variation of S to determine the Euler-Lagrange equations,

$$\delta S = 0 \longleftrightarrow \frac{\delta S}{\delta \phi} = 0 \ ,$$

we need

$$\int_{t_0}^{t_1} dt \int_{\partial^3 V} d\sigma_i \, \left(\frac{\delta S}{\delta \phi_i}\delta\phi + \partial_j(\frac{\delta S}{\delta \phi_{ij}}\delta\phi) \right) + \left[\int_{^3V} d^3x \, (\frac{\delta S}{\delta \phi_{00}}\delta\dot{\phi}) \right]_{t=t_0}^{t=t_1} = 0. \tag{21}$$

Note in the last term the presence of $\delta\dot{\phi}$. This term complicates the setting of the boundary conditions for the fields. Formula (21) —or better, formula (20), before the cancellations originated from the variational principle— is responsible for the expression (11) obtained for the Einstein-Hilbert action. Things become a lot easier if we assume our Lagrangian not to contain second order time derivatives, which is the case for $\mathcal{L}_K$. We will continue with this assumption.

4.1 Second order Lagrangians with no second order time derivatives

This implies that now L is $L[\phi, \dot{\phi}]$ only. In this case, (21) simplifies to

$$\int_{t_0}^{t_1} dt \int_{\partial^3 V} d\sigma_i \, \left(\frac{\delta S}{\delta \phi_i}\delta\phi + \partial_j(\frac{\delta S}{\delta \phi_{ij}}\delta\phi) \right) = 0, \tag{22}$$

and since the —finite— range for the time integration is arbitrary, we end up with

$$\int_{\partial^3 V} d\sigma_i \, \left(\frac{\delta S}{\delta \phi_i}\delta\phi + \partial_j(\frac{\delta S}{\delta \phi_{ij}}\delta\phi) \right) = 0. \tag{23}$$

Condition (23) must be fulfilled, *at any time*, by the fields and its variations in order to comply with the variational principle. Two restrictions originate from (23):

- A restriction on the space of field configurations —spatial boundary conditions for the fields.
- Consequently, a restriction on the allowed variations so that the space of field configurations is preserved under such variations.

Both restrictions altogether must imply (23). Obviously the simplest restrictions one can think of are to fix the values of the fields at the spatial boundary so that its variations at the boundary vanish (maybe not all the fields need to be fixed at the spatial boundary, because some of the functional derivatives of S in (23) may vanish identically). This trivially complies with (23).

The restrictions set on the space of field configurations are subject to a consistency test: they must be compatible with the equations of motion,

$$\frac{\delta S}{\delta \phi} = 0. \tag{24}$$

This compatibility is generally nontrivial, and may cause the rejection of some boundary conditions when they do not agree with the dynamics (24) [1]. To implement the compatibility of (23) with (24), that must hold at any time, we can run, for a fixed time that can be taken as the initial time —when the initial conditions are set—, a Dirac-like algorithm of stabilisation of constraints. Then we can end up, in principle, with secondary boundary conditions, tertiary, etc., that play the role of gauge fixing constraints. We refer to [30] for a form of the standard Dirac algorithm for obtaining of the secondary, tertiary, etc., constraints, that makes no mention to tangency conditions. The idea that boundary conditions must be treated as Dirac constraints is clear from the developments above and has been put forward in [31].

We have all things ready for the study of the implementation of the Noether symmetries for $\mathcal{L}_K$. Before finishing this section, let us mention that the first line of equation (20) —or the more general version (15)—, together with the variation of $\mathcal{L}_K$ in (13), dictates the results:

$$\frac{\delta S_K}{\delta g_{ab,\mu}} = P^{ab}(\mu) \tag{25}$$

for $a, b = 0, 1, 2, 3$ *except* μ, and

$$\frac{\delta S_K}{\delta g_{AB,\mu\nu}} + \frac{\delta S_K}{\delta g_{AB,\nu\mu}} = -\alpha(\mu\nu)\sqrt{|\gamma(\mu\nu)|}\gamma^{AB}(\mu\nu) \tag{26}$$

for $A, B = 0, 1, 2, 3$ *except* μ, ν. Note that (26) is in agreement with (14). These results will be used in section 5.

[1] We leave aside the case when the boundary is an interface between two different physical regimes.

4.2 Noether currents and charges

We first continue developing the general theory with the same assumptions of the preceding section: L is $L[\phi, \dot{\phi}]$ and $\mathcal{L}$ is $\mathcal{L}(\phi, \dot{\phi}, \phi_i, \dot{\phi}_i, \phi_{ij})$

Now consider the Noether case. Suppose that our Lagrangian satisfies, under a certain variation δ_D (not necessarily complying with (23)),

$$\delta_D \mathcal{L} = \partial_\mu F^\mu \ , \tag{27}$$

for certain functions F^μ. From the first line of (20),

$$\delta_D \mathcal{L} = \frac{\delta S}{\delta \phi} \delta_D \phi + \partial_\mu (\frac{\delta S}{\delta \phi_\mu} \delta_D \phi) + \partial_\nu \partial_\mu (\frac{\delta S}{\delta \phi_{\mu\nu}} \delta_D \phi),$$

we get

$$\frac{\delta S}{\delta \phi} \delta_D \phi = \partial_\mu \left(F^\mu - \frac{\delta S}{\delta \phi_\mu} \delta_D \phi - \partial_\nu (\frac{\delta S}{\delta \phi_{\mu\nu}} \delta_D \phi) \right) , \tag{28}$$

which identifies the current

$$J^\mu := F^\mu - \frac{\delta S}{\delta \phi_\mu} \delta_D \phi - \partial_\nu (\frac{\delta S}{\delta \phi_{\mu\nu}} \delta_D \phi) + \partial_\nu A^{\mu\nu}, \tag{29}$$

as an on shell conserved object,

$$\partial_\mu J^\mu \underset{\text{(on shell)}}{=} 0 \ . \tag{30}$$

Note that we have included in J^μ the unavoidable ambiguity of the addition of the divergence of an arbitrary antisymmetric object $A^{\mu\nu}$. When searching for a conserved charge, it will prove useful to take into account this ambiguity. Let us integrate (30) along the spatial 3-volume,

$$\partial_0 \int_{^3V} d^3x \ J^0 + \int_{\partial^3 V} d\sigma_i \ J^i \underset{\text{(on shell)}}{=} 0, \tag{31}$$

so the charge

$$Q := \int_{^3V} d^3x \ J^0 \tag{32}$$

will be conserved on shell if

$$\int_{\partial^3 V} d\sigma_i \ J^i \underset{\text{(on shell)}}{=} 0.$$

It is advantageous to take, in J^μ,

$$A^{i0} = -A^{0i} = \frac{\delta S}{\delta \phi_{i0}} \delta_D \phi + B^{i0} \ , \quad A^{ij} = B^{ij} \ ,$$

with $B^{\mu\nu}$ another arbitrary antisymmetric object. Then the components of the current are ,

$$J^0 = F^0 - \frac{\delta S}{\delta \dot{\phi}} \delta_D \phi - \partial_i (\frac{\delta S}{\delta \dot{\phi}_i} \delta_D \phi - B^{0i})$$

and

$$J^i = F^i - \frac{\delta S}{\delta \phi_i} \delta_D \phi - \partial_j (\frac{\delta S}{\delta \phi_{ij}} \delta_D \phi) + \partial_\nu B^{i\nu} \ .$$

Thus,

$$\int_{\partial^3 V} d\sigma_i \; J^i = \int_{\partial^3 V} d\sigma_i \; \left(F^i - \frac{\delta S}{\delta \phi_i} \delta_D \phi - \partial_j (\frac{\delta S}{\delta \phi_{ij}} \delta_D \phi) + \partial_\nu B^{i\nu}) \right) \; ,$$

and now (23) comes into play: the addition of the second and third terms in the right side vanish *if the field configurations and δ_D comply with (23).* Here we realize the relevance of the boundary conditions when considering the conservation of charge. Form now on we assume that the variational principle has restricted our space of field configurations and the allowed variations δ_D defined on it in such way that (23) is satisfied. Then the charge conservation (32) has the condition

$$\int_{\partial^3 V} \; d\sigma_i (F^i + \partial_\nu B^{i\nu})) \underset{\text{(on shell)}}{=} 0$$

whereas the charge itself takes the form

$$Q := \int_{^3 V} d^3x \; J^0 = \int_{^3 V} d^3x \; \left(F^0 - \frac{\delta S}{\delta \dot{\phi}} \delta_D \phi - \partial_i (\frac{\delta S}{\delta \dot{\phi}_i} \delta_D \phi - B^{0i}) \right) .$$

The presence of the object $B^{\mu\nu}$ may be endowed to the ambiguity of F^μ in (27), under additions of the divergence of an arbitrary antisymmetric object. Absorbing $\partial_\nu B^{i\nu}$ within F^i and $\partial_i B^{0i}$ within F^0, we arrive at the following

Proposition 1 *Given a second order Lagrangian with no second order time derivatives. If*

1. *the space of field configurations —including spatial boundary conditions for the fields—, together with the allowed variations defined on it, complies with (23),*

2. *the boundary conditions imposed on the field configurations are compatible with the equations of motion (24),*

3. *there is an allowed variation δ_D such that $\delta_D \mathcal{L} = \partial_\mu F^\mu$ for some F^μ,*

then, the charge

$$Q = \int_{^3 V} d^3x \; (F^0 - \frac{\delta S}{\delta \dot{\phi}} \delta_D \phi) - \int_{\partial^3 V} d\sigma_i \; (\frac{\delta S}{\delta \dot{\phi}_i} \delta_D \phi) \tag{33}$$

is conserved if and only if the condition

$$\int_{\partial^3 V} d\sigma_i \; F^i \underset{\text{(on shell)}}{=} 0 \; , \tag{34}$$

holds.

For infinite boundaries, as in the case of the limit $r \to \infty$ for asymptotically flat spaces, (34) is understood as

$$\lim_{r \to \infty} \int_{\partial^3 V} d\sigma_i \; F^i \underset{\text{(on shell)}}{=} 0 \; . \tag{35}$$

Note the generic form of the charge —conserved if the conditions of proposition 1 hold— in (33): a bulk term and a boundary term, the existence of this last term having its origin in the dependence of the Lagrangian on the second order derivatives.

It is worth to remark that the Noether current conservation (30) stemming from (28) is a local property *only* of the equations of motion, and it is totally impervious to the modifications of the action by boundary terms. The Noether current is local, and for its definition (29) it is completely irrelevant whether the variation δ_D producing (27) is an allowed variation —in the sense of (23)— or not. Let the action be S_{EH}, $S_{\mathbf{\Gamma}}$ or S_K —each leading to different boundary conditions—, the current conservation equation (30) will always be the same (but remember: there is an ambiguity in J^μ, to wit, the addition of a divergence of an arbitrary antisymmetric object). What then is the relevance of using one action or another? the answer is that the boundary terms in the action are indeed relevant to set the conditions (23). Selecting the action entails a selection of the space of field configurations and the allowed variations on it —though maybe not in a unique way. It is obvious then, looking at the proposition above, that the charge conservation crucially depends on the selected action.

If the current conservation (30) holds but the conditions of Proposition 1 are not satisfied, then the equation (31) will express the typical conservation equation that balances the rate of change in time of the total charge with the flux of current through the boundary.

4.3 Energy and (pre)symplectic structure

Here we continue with a second order Lagrangian $\mathcal{L}$ with no second order time derivatives. Since there is no explicit dependence of $\mathcal{L}$ on the coordinates, $\delta_D\phi = \dot{\phi}\delta t$, with δt an infinitesimal constant, is a Noether symmetry with $F^\mu = \delta^\mu_0 \mathcal{L}\delta t$. Since $F^i = 0$, the conservation of (33) will always hold *as long as conditions 1) and 2) in Proposition 1 are satisfied.* With this proviso, the conserved quantity (33) is (up to a sign) the energy:

$$E = \int_{^3V} d^3x \; (\frac{\delta S}{\delta\dot{\phi}}\dot{\phi} - \mathcal{L}) + \int_{\partial^3 V} d\sigma_i \; (\frac{\delta S}{\delta\dot{\phi}_i}\dot{\phi}) \; , \tag{36}$$

that can also be expressed in terms of the Lagrangian functional,

$$E = \int_{^3V} d^3x \; (\frac{\delta L}{\delta\dot{\phi}}\dot{\phi}) + \int_{\partial^3 V} d\sigma_i \; (\frac{\delta L}{\delta\dot{\phi}_i}\dot{\phi}) - L[\phi, \; \dot{\phi}].$$

Expression (36) is the formula for the energy for second order Lagrangians with no second order time derivatives. It generalises the common expression $E = \hat{p}_k\dot{q}^k - L(q,\dot{q})$, corresponding to the Legendre transformation in mechanics, where

$$\hat{p}_k := \frac{\partial L}{\partial \dot{q}^k}$$

is the pullback to tangent space of the momentum in cotangent space (phase space) through the Legendre map. Some interesting consequences may be drawn from this generalisation. Summation for an index k in mechanics becomes in field theory integration for the space coordinates plus summation for all the fields. Using this mechanical analogy, the role of the pullback $\hat{p}_k dq^k$ of the Liouville one-form is now played by

$$\int_{^3V} d^3x \; (\frac{\delta S}{\delta\dot{\phi}}\delta\phi) + \int_{\partial^3 V} d\sigma_i \; (\frac{\delta S}{\delta\dot{\phi}_i}\delta\phi) \; ,$$

which can indeed be called the pullback of the Liouville form for this type of field theory. The pullback $d\hat{p}_k \wedge dq^k$ to tangent space of the symplectic two-form in phase space now becomes

$$\hat{\Omega} = \int_{^3V} d^3x \; (\delta \frac{\delta S}{\delta \dot{\phi}} \wedge \delta\phi) + \int_{\partial^3 V} d\sigma_i \; (\delta \frac{\delta S}{\delta \dot{\phi}_i} \wedge \delta\phi) \; . \tag{37}$$

This structure can be symplectic (closed and maximal rank) or presymplectic (non maximal rank) depending on the presence of gauge symmetries in the theory.

Expression (37) also means that each canonical momentum has now two components: the bulk component and the boundary component. Their pullbacks to tangent space are, respectively,

$$p_{(\mathrm{bulk})} = \frac{\delta S}{\delta \dot{\phi}} \tag{38}$$

and

$$p^i_{(\mathrm{boundary})} = \frac{\delta S}{\delta \dot{\phi}_i} \; . \tag{39}$$

5 Application to $\mathcal{L}_{\mathcal{K}}$: asymptotically flat spaces

Here we follow Faddeev approach [26]. Asymptotically flat spacetimes correspond to physical situations where the gravitating masses and matter fields at finite times are effectively concentrated in a finite region of space. Our spacetime will be a topologically simple manifold whose points can be parametrised by a system of four coordinates x^μ, $-\infty < x^\mu < \infty$, such that, in the limit $r \to \infty$ ($r^2 = (x^1)^2 + (x^2)^2 + (x^3)^2$) for finite time $t = x^0$, the metric components satisfy the asymptotic conditions

$$\begin{aligned} g_{\mu\nu} &= \eta_{\mu\nu} + O(\frac{1}{r}), \quad \partial_\sigma g_{\mu\nu} = O(\frac{1}{r^2}), \\ \partial_{\sigma\rho} g_{\mu\nu} &= O(\frac{1}{r^3}), \quad \cdots, \partial^{(m)} g_{\mu\nu} = O(\frac{1}{r^{(m+1)}}). \end{aligned} \tag{40}$$

Conditions (40) amount to a partial gauge fixing, for the only acceptable changes of coordinates will be from now on those that preserve (40). To this end, for an infinitesimal change $x^\mu \to x'^\mu = x^\mu - \epsilon^\mu(x)$, we take ϵ^μ to behave as

$$\begin{aligned} \epsilon^\mu &= \omega^\mu{}_\nu x^\nu + a^\mu + O(\frac{1}{r}), \\ \partial_\nu \epsilon^\mu &= \omega^\mu{}_\nu + O(\frac{1}{r^2}) \\ \partial_\nu \partial_\sigma \epsilon^\mu &= O(\frac{1}{r^{2+\alpha}}), \; \alpha > 0 \\ &\cdots \\ \partial^{(m)} \epsilon^\mu &= O(\frac{1}{r^{m+\alpha}}), \; \alpha > 0 \; . \end{aligned} \tag{41}$$

$\omega^\mu{}_\nu$ is an infinitesimal Lorentz parameter, $\eta_{\rho\mu}\omega^\mu{}_\nu + \eta_{\nu\mu}\omega^\mu{}_\rho = 0$, and a^μ is an infinitesimal translation of the coordinates.

Note that, under an infinitesimal diffeomorphism (41),

$$\begin{aligned} \delta_D g_{\mu\nu} &= \epsilon^\rho g_{\mu\nu,\rho} + g_{\mu\rho}\epsilon^\rho_{,\nu} + g_{\rho\nu}\epsilon^\rho_{,\mu} \\ &= \mathcal{O}(r)\times\mathcal{O}(\frac{1}{r^2}) + \Big((\eta_{\mu\rho}+\mathcal{O}(\frac{1}{r}))(\omega^\rho{}_\nu+\mathcal{O}(\frac{1}{r^2}))\Big) \\ &+ \Big((\eta_{\rho\nu}+\mathcal{O}(\frac{1}{r}))(\omega^\rho{}_\mu+\mathcal{O}(\frac{1}{r^2}))\Big) \\ &= \mathcal{O}(\frac{1}{r}), \end{aligned} \tag{42}$$

in agreement with the asymptotic behavior (40). Also $\delta_D \partial^{(m)} g_{\mu\nu} = \mathcal{O}(\frac{1}{r^{(m+1)}})$.

Let us check that (40) and (41) imply the boundary conditions (23). The integration in (23) is, for $\mathcal{L}_K$,

$$\begin{aligned} B.C. &:= \int_{\partial^3 V} d\sigma_i \, \Big(\frac{\delta S_K}{\delta\phi_i}\delta\phi + \partial_j(\frac{\delta S_K}{\delta\phi_{ij}}\delta\phi)\Big) = \int_{\partial^3 V} d\sigma_i \, \Big(\frac{\delta S_K}{\delta\phi_i}\delta\phi + \frac{1}{2}\partial_j(\frac{\delta^c S_K}{\delta\phi_{ij}}\delta\phi)\Big) \\ &= \int_{\partial^3 V} d\sigma_i \, \Big(P^{ab}(i)\delta g_{ab} - \frac{1}{2}\partial_j(\alpha(ij)\sqrt{|\gamma(ij)|}\gamma^{AB}(ij)\delta g_{AB})\Big) \\ &= \int_{\partial^3 V} d\sigma_i \, \Big(P^{ab}(i)\delta g_{ab} - \partial_j(\alpha(ij)\delta\sqrt{|\gamma(ij)|})\Big) \ , \end{aligned} \tag{43}$$

where we have used (25) and (26).

Since, when $r\to\infty$, the area of the boundary $\partial^3 V$ grows as $\mathcal{O}(r^2)$, the asymptotic behavior of (43) is

$$\begin{aligned} \lim_{r\to\infty} B.C. &= \lim_{r\to\infty}\int_{\partial^3 V} d\sigma_i \, \Big(P^{ab}(i)\delta g_{ab} - \partial_j(\alpha(ij))\delta\sqrt{|\gamma(ij)|} - \alpha(ij)\partial_j(\delta\sqrt{|\gamma(ij)|})\Big) \\ &= \lim_{r\to\infty}\mathcal{O}(r^2)\Big(\mathcal{O}(\frac{1}{r^2})\times\mathcal{O}(\frac{1}{r}) - \mathcal{O}(\frac{1}{r^2})\times\mathcal{O}(\frac{1}{r}) - \mathcal{O}(\frac{1}{r})\times\mathcal{O}(\frac{1}{r^2})\Big) \\ &= \lim_{r\to\infty}\mathcal{O}(\frac{1}{r}) = 0 \ , \end{aligned} \tag{44}$$

in agreement with (23). It remains now to check that there is consistency of the conditions (40) with the equations of motion (24) —for finite times. This can be verified using the Lagrangian equations of motion for $\mathcal{L}_K$ —or $\mathcal{L}_{EH}$. The conditions of Proposition 1 are therefore satisfied.

Let us apply to $\mathcal{L}_K$ the results of the preceding section. The bulk momentum is obtained from (25),

$$\frac{\delta S_K}{\delta \dot{g}_{ij}} = P^{ij}(0) =: P^{ij} \ , \tag{45}$$

for $i,j=1,2,3$; and the boundary momentum form (26),

$$\frac{\delta S_K}{\delta \dot{g}_{AB,i}} = -\alpha(0i)\sqrt{|\gamma(0i)|}\gamma^{AB}(0i) \ , \tag{46}$$

for $A,B=1,2,3$ *except* i. Let us find the Liouville and the presymplectic forms. The Liouville form is

$$\hat{p}_k dq^k = \int_{^3V} d^3x \, (\frac{\delta S_K}{\delta\dot\phi}\delta\phi) + \int_{\partial^3 V} d\sigma_i \, (\frac{\delta S_K}{\delta\dot\phi_i}\delta\phi)$$

$$
\begin{aligned}
&= \int_{^3V} d^3x(P^{ij}\delta g_{ij}) - \int_{\partial^3 V} d\sigma_i\ (\alpha(0i)\sqrt{|\gamma(0i)|}\gamma^{AB}(0i)\delta g_{AB}) \\
&= \int_{^3V} d^3x(P^{ij}\delta g_{ij}) - 2\int_{\partial^3 V} d\sigma_i\ (\alpha(0i)\delta\sqrt{|\gamma(0i)|})\ ,
\end{aligned} \tag{47}
$$

and the presymplectic form (37) is, accordingly,

$$
\begin{aligned}
\hat\Omega &= \int_{^3V} d^3x(\delta\frac{\delta S_K}{\delta\dot\phi}\wedge\delta\phi) + \int_{\partial^3 V} d\sigma_i\ (\delta\frac{\delta S_K}{\delta\dot\phi_i}\wedge\delta\phi) \\
&= \int_{^3V} d^3x\ (\delta P^{ij}\wedge\delta g_{ij}) - 2\int_{\partial^3 V} d\sigma_i\ (\delta\alpha(0i)\wedge\delta\sqrt{|\gamma(0i)|})\ .
\end{aligned} \tag{48}
$$

This result was obtained in [22, 23] using the theory of symplectic relations [32, 33, 34]; here it has been derived as a particular case of (37).

According to the remarks produced at the end of subsection 4.2, since $\mathcal{L}_K$ differs from the scalar density Lagrangian $\mathcal{L}_{EH}$ by boundary terms, it is already guaranteed that Noether currents exist for all diffeomorphism symmetries. The point is that any Lagrangian $\mathcal{L}_{\rm any}$ differing from $\mathcal{L}_{EH}$, a scalar density, by a divergence term,

$$
\mathcal{L}_{\rm any} = \mathcal{L}_{EH} + \partial_\mu D^\mu\ ,
$$

transforms, under an infinitesimal diffeomorphism δ_D generated by the vector field $\epsilon^\mu\partial_\mu$ (not necessarily satisfying (41)), as

$$
\delta_D\mathcal{L}_{\rm any} = \partial_\mu(\epsilon^\mu\mathcal{L}_{EH} + \delta_D D^\mu).
$$

5.1 The energy

Let us consider now the energy for a theory described by $\mathcal{L}_K$ plus matter terms in an asymptotically flat spacetime. We will assume that these matter terms are described by a first order scalar density matter Lagrangian that together with $\mathcal{L}_K$ define the total Lagrangian, and that the couplings with the metric are nonderivative. We will also assume that the matter terms satisfy the appropriate conditions at the boundary in order to comply with the variational principle. We will first work with pure gravity and at the end consider the changes introduced by the presence of matter. So we continue with the Lagrangian $\mathcal{L}_K$.

Expression (36) gives, for the energy density $\mathcal{E}$ ($E = \int_{^3V} d^3x\ \mathcal{E}$) ,

$$
\begin{aligned}
\mathcal{E} &= \frac{\delta S}{\delta\dot\phi}\dot\phi - \mathcal{L} + \partial_i(\frac{\delta S}{\delta\dot\phi_i}\dot\phi) \\
&= P^{ij}\dot g_{ij} - \mathcal{L}_K - 2\partial_i(\alpha(0i)\partial_0\sqrt{|\gamma(0i)|}).
\end{aligned} \tag{49}
$$

Introducing (12) and recalling $P^{ij}g_{ij} = 2\sqrt{|\gamma(0)|}K(0)$,

$$
\begin{aligned}
\mathcal{E} &= P^{ij}\dot g_{ij} - \sqrt{|g|}R - 2\partial_i(\sqrt{|\gamma(i)|}K(i)) - \partial_0(P^{ij}g_{ij}) \\
&+ \partial_i\partial_j(\sqrt{|\gamma(ij)|}\alpha(ij)) + 2\partial_i\partial_0(\sqrt{|\gamma(0i)|}\alpha(0i)) - 2\partial_i(\alpha(0i)\partial_0\sqrt{|\gamma(0i)|}) \\
&= -\dot P^{ij}g_{ij} - \sqrt{|g|}R - 2\partial_i(\sqrt{|\gamma(i)|}K(i)) \\
&+ \partial_i\partial_j(\sqrt{|\gamma(ij)|}\alpha(ij)) + 2\partial_i(\sqrt{|\gamma(0i)|}\partial_0\alpha(0i)).
\end{aligned} \tag{50}
$$

Let us now rewrite some terms in (50). We show in the appendix, using the methods of [21], that

$$-\dot{P}^{ij}g_{ij}+2\partial_i(\sqrt{|\gamma(0i)|}\partial_0\alpha(0i))=2\sqrt{|g|}R^0_0-2\partial_i(\sqrt{|g|}\gamma^{0\mu}(i)\,\mathbf{\Gamma}^i_{0\mu}) \tag{51}$$

where R^0_0 is a component of the Ricci tensor. On the other hand, using (10), another piece in (50) can be given a more convenient expression

$$\begin{aligned}\sqrt{|\gamma(ij)|}\partial_j\alpha(ij) &= \frac{1}{2}\sqrt{|g|}\left(g^{ii}\partial_j(\frac{g^{ij}}{g^{ii}})+g^{jj}\partial_j(\frac{g^{ij}}{g^{jj}})\right)\\ &= -\sqrt{|g|}(\mathbf{\Gamma}^i_{j\mu}\gamma^{\mu j}(i)+\mathbf{\Gamma}^j_{j\mu}\gamma^{\mu i}(j))\ ,\end{aligned} \tag{52}$$

where $\nabla_\mu g^{\rho\sigma}=0$ has been used in the last equality. Also, trivially,

$$\sqrt{|\gamma(i)|}K(i)=-\sqrt{|g|}\gamma^{\mu\nu}(i)\mathbf{\Gamma}^i_{\mu\nu}. \tag{53}$$

All together,

$$\begin{aligned}\mathcal{E} &= \sqrt{|g|}(2R^0_0-R)+2\partial_i(\sqrt{|g|}\gamma^{\mu\nu}(i)\mathbf{\Gamma}^i_{\mu\nu})-\partial_i\left(\sqrt{|g|}(\mathbf{\Gamma}^i_{j\mu}\gamma^{\mu j}(i)+\mathbf{\Gamma}^j_{j\mu}\gamma^{\mu i}(j))\right)\\ &+ \partial_i(\alpha(ij)\partial_j\sqrt{|\gamma(ij)|})-2\partial_i(\sqrt{|g|}\gamma^{0\mu}(i)\mathbf{\Gamma}^i_{0\mu})\\ &= 2G^0_0+\partial_i(\alpha(ij)\partial_j\sqrt{|\gamma(ij)|})+\partial_i\left(\sqrt{|g|}(\mathbf{\Gamma}^i_{j\mu}\gamma^{\mu j}(i)-\mathbf{\Gamma}^j_{j\mu}\gamma^{\mu i}(j))\right)\ .\end{aligned} \tag{54}$$

The first term in (54), a component of the Einstein tensor, vanishes on shell —it is a Lagrangian constraint, part of the equations of motion. The total energy on shell is

$$\begin{aligned}E &\underset{\text{(on shell)}}{=} \int_{^3V}d^3x\ \mathcal{E}|_{\text{on shell}}\\ &= \lim_{r\to\infty}\int_{\partial^3V}d\sigma_i\ \left(\alpha(ij)\partial_j\sqrt{|\gamma(ij)|}+\sqrt{|g|}(\mathbf{\Gamma}^i_{j\mu}\gamma^{\mu j}(i)-\mathbf{\Gamma}^j_{j\mu}\gamma^{\mu i}(j))\right)\ .\end{aligned} \tag{55}$$

Considering the asymptotic behavior (40), the contribution of the first term in (55) vanishes because $\alpha(ij)=\mathcal{O}(\frac{1}{r})$. The second term contributes

$$E\underset{\text{(on shell)}}{=}\lim_{r\to\infty}\int_{\partial^3V}d\sigma_i\ (\mathbf{\Gamma}^i_{jj}-\mathbf{\Gamma}^j_{ij})=\lim_{r\to\infty}\int_{\partial^3V}d\sigma_i\ (\partial_jg_{ij}-\partial_ig_{jj})\ , \tag{56}$$

which is the ADM [24] energy.

The inclusion of matter, with the restrictions set at the beginning of this section, will change the term G^0_0 to $G^0_0-8\pi T^0_0$, where T^0_0 is a component of the energy momentum tensor for the matter fields. Now $G^0_0-8\pi T^0_0$ is a constraint that vanishes in virtue of Einstein equations. The asymptotic contribution in (56) remains the same.

5.2 Other Noether charges for $\mathcal{L}_K$

Space translations in the j direction (parameter a^j in (41)) define for $\mathcal{L}_K$ the quantity $F^i_{(j)}$ in (27),

$$F^i_{(j)}=\delta^i_j\mathcal{L}_Ka^j$$

that should satisfy (34) in Proposition 1. The $r \to \infty$ behavior of $\mathcal{L}_K$ can be easily deduced from the expression (14). The terms quadratic in the derivatives of the metric behave as $\mathcal{O}(\frac{1}{r^2}) \times \mathcal{O}(\frac{1}{r^2}) = \mathcal{O}(\frac{1}{r^4})$. The term $\alpha(\mu\nu)\partial_\mu\partial_\nu(\sqrt{|\gamma(\mu\nu)|})$ has

$$\alpha(\mu\nu) \underset{r\to\infty}{\simeq} \mathcal{O}(\frac{1}{r}) \ ,$$

and

$$\partial_\mu\partial_\nu(\sqrt{|\gamma(\mu\nu)|}) \underset{r\to\infty}{\simeq} \mathcal{O}(\frac{1}{r^3}) \ .$$

Therefore

$$\mathcal{L}_K \underset{r\to\infty}{\simeq} \mathcal{O}(\frac{1}{r^4}) \ ,$$

and so,

$$\lim_{r\to\infty} \int_{\partial^3 V} d\sigma_i \ \mathcal{L}_K = 0 \ . \tag{57}$$

This proves that the translational Noether symmetry leads to conserved quantities, the momenta.

This result can be generalised: any diffeomorphism satisfying (41) has a conserved charge. To prove it we need to compute F^μ in (27) for an infinitesimal diffeomorphism δ_D generated by $\epsilon^\mu\partial_\mu$. This computation is involved because $\mathcal{L}_K$ is not a scalar density. The definition (12) obviously gives

$$\delta_D\mathcal{L}_K = \partial_\mu(\epsilon^\mu\mathcal{L}_{EH}) + 2\partial_\mu\left(\delta_D(\sqrt{|\gamma(\mu)|}K(\mu))\right) - \partial_\mu\partial_\nu\left(\delta_D(\sqrt{|\gamma(\mu\nu)|}\alpha(\mu\nu))\right) \ . \tag{58}$$

but the behavior in the limit $r \to \infty$ of the first term

$$\epsilon^\mu\mathcal{L}_{EH} \underset{r\to\infty}{\simeq} \mathcal{O}(\frac{1}{r^2}) \ ,$$

(with $\epsilon^\mu \underset{r\to\infty}{\simeq} \mathcal{O}(r)$) produces terms different from zero in (35). In fact this behavior is corrected by other terms in (58) but to single out these contributions is cumbersome. It is much more convenient to express $\mathcal{L}_K$ as the sum of the "gamma-gamma" Lagrangian $\mathcal{L}_{\mathbf{\Gamma}}$ plus divergences, because the behavior of $\mathcal{L}_{\mathbf{\Gamma}}$ in the limit $r \to \infty$ is much better than that of $\mathcal{L}_{EH}$,

$$\epsilon^\mu\mathcal{L}_{\mathbf{\Gamma}} \underset{r\to\infty}{\simeq} \mathcal{O}(\frac{1}{r^3}) \ .$$

So let us proceed to relate $\mathcal{L}_K$ with $\mathcal{L}_{\mathbf{\Gamma}}$. Using (53), a trivial extension of (52) to indices $\mu\nu$,

$$\begin{aligned} \partial_\mu\left(\sqrt{|\gamma(\mu\nu)|}\partial_\nu\alpha(\mu\nu)\right) &= -\partial_\mu\left(\sqrt{|g|}(\mathbf{\Gamma}^\mu_{\nu\sigma}\gamma^{\nu\sigma}(\mu) + \mathbf{\Gamma}^\nu_{\nu\sigma}\gamma^{\sigma\mu}(\nu))\right) \\ &= 2\partial_\mu\left(\sqrt{|\gamma(\mu)|}K(\mu)\right) + \partial_\mu\left(\sqrt{|g|}(\mathbf{\Gamma}^\mu_{\nu\sigma}\gamma^{\nu\sigma}(\mu) - \mathbf{\Gamma}^\nu_{\nu\sigma}\gamma^{\sigma\mu}(\nu))\right) \end{aligned} \tag{59}$$

allows to write $\mathcal{L}_K$ in the equivalent form

$$\mathcal{L}_K = \sqrt{|g|}R - \partial_\mu\left(\alpha(\mu\nu)\partial_\nu\sqrt{|\gamma(\mu\nu)|}\right) - \partial_\mu\left(\sqrt{|g|}(\mathbf{\Gamma}^\mu_{\nu\sigma}\gamma^{\nu\sigma}(\mu) - \mathbf{\Gamma}^\nu_{\nu\sigma}\gamma^{\sigma\mu}(\nu))\right) \ . \tag{60}$$

On the other hand, the "gamma-gamma" Lagrangian (2) can be written as

$$\mathcal{L}_{\mathbf{\Gamma}} = \sqrt{|g|}R - \partial_\mu\left(\sqrt{|g|}(\mathbf{\Gamma}^\mu_{\nu\sigma}g^{\nu\sigma} - \mathbf{\Gamma}^\nu_{\nu\sigma}g^{\sigma\mu})\right) \ , \tag{61}$$

and therefore, using (4),

$$\begin{aligned}\mathcal{L}_K &= \mathcal{L}_{\mathbf{\Gamma}} - \partial_\mu\left(\alpha(\mu\nu)\partial_\nu\sqrt{|\gamma(\mu\nu)|}\right)\\ &+ \partial_\mu\left(\sqrt{|g|}(\eta(\mu)n^\sigma(\mu)n^\nu(\mu)\mathbf{\Gamma}^\mu_{\nu\sigma} - \eta(\nu)n^\sigma(\nu)n^\mu(\nu)\mathbf{\Gamma}^\nu_{\nu\sigma})\right) .\end{aligned} \quad (62)$$

We continue with δ_D being an infinitesimal diffeomorphism generated by $\epsilon^\mu\partial_\mu$. Computing $\delta_D\mathcal{L}_{\mathbf{\Gamma}}$ is a simple thing [26] because the deviation of $\mathcal{L}_{\mathbf{\Gamma}}$ from the scalar density behavior is due exclusively to the two connexion coefficients present in the divergence term in (61). Since the deviation of the connexion $\mathbf{\Gamma}^\mu_{\nu\sigma}$ from a tensorial behavior is an additive term $\epsilon^\mu_{,\nu\sigma}$, we can write

$$\delta_D\mathcal{L}_{\mathbf{\Gamma}} = \partial_\mu\left(\epsilon^\mu\mathcal{L}_{\mathbf{\Gamma}} - \sqrt{|g|}g^{\nu\sigma}\epsilon^\mu_{,\nu\sigma} + \sqrt{|g|}g^{\mu\sigma}\epsilon^\nu_{,\nu\sigma}\right) ,$$

and thus, for $\mathcal{L}_K$,

$$\begin{aligned}\delta_D\mathcal{L}_K &= \partial_\mu\Bigg(\epsilon^\mu\mathcal{L}_{\mathbf{\Gamma}} - \sqrt{|g|}g^{\nu\sigma}\epsilon^\mu_{,\nu\sigma} + \sqrt{|g|}g^{\mu\sigma}\epsilon^\nu_{,\nu\sigma} - \delta_D\left(\alpha(\mu\nu)\partial_\nu\sqrt{|\gamma(\mu\nu)|}\right)\\ &+ \delta_D\left(\sqrt{|g|}(\eta(\mu)n^\sigma(\mu)n^\nu(\mu)\mathbf{\Gamma}^\mu_{\nu\sigma} - \eta(\nu)n^\sigma(\nu)n^\mu(\nu)\mathbf{\Gamma}^\nu_{\nu\sigma})\right)\Bigg) .\end{aligned} \quad (63)$$

Now, from (63), we deduce the object F^i that needs to pass the test (35),

$$\begin{aligned}F^i &= \epsilon^i\mathcal{L}_{\mathbf{\Gamma}} - \sqrt{|g|}g^{\nu\sigma}\epsilon^i_{,\nu\sigma} + \sqrt{|g|}g^{i\sigma}\epsilon^\nu_{,\nu\sigma} - \delta_D\left(\alpha(i\nu)\partial_\nu\sqrt{|\gamma(i\nu)|}\right)\\ &+ \delta_D\left(\sqrt{|g|}(\eta(i)n^\sigma(i)n^\nu(i)\mathbf{\Gamma}^i_{\nu\sigma} - \eta(\nu)n^\sigma(\nu)n^i(\nu)\mathbf{\Gamma}^\nu_{\nu\sigma})\right) .\end{aligned} \quad (64)$$

Now it is easy to check that each additive term in (64), under the conditions (40) and (41), guarantees (35) and therefore

$$\lim_{r\to\infty}\int_{\partial^3 V} d\sigma_i\ F^i = 0 ,$$

thus proving that all diffeomorphisms satisfying (41) give Noether conserved charges for asymptotically flat spaces defined by the conditions (40). Note in particular the existence of conserved charges associated with the Poincaré transformations included in (41). In this sense, (56) is just an example of the computation —the total energy— of one of the ten conserved charges corresponding to the Poincaré group.

6 Conclusions

In this paper we have developed the formal theory for the variational principle and the Noether symmetries for second order Lagrangians in field theories with boundaries, with special emphasis in Lagrangians with no second order time derivatives. These developments lead to general expressions for some geometric objects, like the pullback to tangent space of the symplectic form in phase space. It is worth to remark that bulk terms and boundary terms contribute to these objects. Also the canonical momenta exhibit both a bulk piece and a boundary piece.

The Noether theorem is discussed for these theories. A clear distinction is made between conservation of currents, that is a local assertion only related to the equations of motion —but not to the possible boundary terms in the action—, and the conservation of charges, where the relation with the boundary terms appearing in the action is made transparent. The reason being that the boundary terms in the action are linked to the boundary conditions to be satisfied by the fields. These results are summarised in a proposition introduced in subsection 4.2. Let us mention also that the Noether charge exhibits a bulk piece and a boundary piece, the boundary piece owing its existence to the second order dependences in the Lagrangian.

This framework is applied to the trace K Lagrangian for General Relativity. The trace K Lagrangian is obtained in section 3 and its applications to asymptotically flat spaces are studied in section 5. We observe that this Lagrangian is not first order in the derivatives but its asymptotic behavior is different from that of the Einstein-Hilbert Lagrangian and resembles that of first order Lagrangians like the gamma-gamma Lagrangian. We prove the conservation of charges for diffeomorphisms that preserve the boundary conditions for the metric tensor and, as a particular case, we obtain the ADM formula for the conserved energy.

It is worth noting that we have based our approach in keeping at any moment the requirements derived from the strict application of the variational principle, including in particular the differentiability of the action functional. These requirements lead to the imposition of boundary conditions on the field configurations. When we examine the conditions for the conservation of possible Noether charges the relevance of these boundary conditions, and hence of the variational principle, becomes transparent.

Let us finally give a short review of the main results obtained in this paper.

1. General Theory

    - In Proposition 1, section 4.2, we have given the necessary and sufficient conditions for the conservation of Noether charges for field theories with boundaries; these field theories being derived from action principles with second order Lagrangians that are free from second order time derivative terms.
    - We have shown that the Noether conserved charges, (33), exhibit a boundary piece that stems from the presence of the one-time-one-space derivative terms in the Lagrangian.
    - We have derived a general formula, (37), for the (pre)symplectic form associated with this type of Lagrangians. This form exhibits a boundary piece that has the same origin as for the Noether charges.
    - Also, due to the presence of the one-time-one-space derivative terms in the Lagrangian, we show that one is naturally led to define a bulk component, (38), as well as a boundary component, (39), for each of the momenta.

2. Trace K Lagrangian

- We have neatly displayed, (14), the dependence of the Trace K Lagrangian on second order spacetime derivatives. We also show that the asymptotic behavior of this Lagrangian is similar to the one of the first order "gamma-gamma" Lagrangian, substantially different from that of the Einstein-Hilbert Lagrangian.
- We have applied our general formulas to compute the (pre)symplectic form associated with the Trace K Lagrangian, (48). Our results coincide with those obtained within the theory of symplectic relations.
- We have given an expression for the energy density, (54), and we have computed the total energy for the asymptotically flat case, obtaining the ADM formula.
- We have shown that, in the asymptotically flat case, all diffeomorphisms satisfying the boundary conditions -this includes the transformations that are asymptotically Poincaré- give Noether conserved charges.

7 Appendix

7.1 Deviation of the transformation of $\mathbf{n}(\mu)$ from the vector behavior

Let δ_D be the infinitesimal diffeomorphism generated by $\epsilon^\mu \partial_\mu$. Our task is to compute $\delta_D n^\nu(\mu)$ for the "vector" $\mathbf{n}(\mu)$whose components, $n^\nu(\mu)$, are defined through

$$n^\mu(\mu) n^\nu(\mu) = \xi(\mu) g^{\mu\nu} \ ,$$

and we know

$$\delta_D g^{\mu\nu} = \epsilon^\sigma \partial_\sigma g^{\mu\nu} - g^{\mu\sigma}\partial_\sigma \epsilon^\nu - g^{\sigma\nu}\partial_\sigma \epsilon^\mu \ .$$

Considering

$$(n^\mu(\mu))^2 = \xi(\mu) g^{\mu\mu} \ ,$$

we obtain

$$\delta_D n^\mu(\mu) = \delta_{\rm naive} n^\mu(\mu) \ ,$$

where we have introduced $\delta_{\rm naive}$ to symbolise a "naive" variation associated with vector behavior, that is,

$$\delta_{\rm naive} n^\nu(\mu) := \epsilon^\sigma \partial_\sigma n^\nu(\mu) - n^\sigma(\mu)\partial_\sigma \epsilon^\nu .$$

Next, to find $\delta_D n^\nu(\mu)$, use

$$n^\nu(\mu) = \xi(\mu) g^{\mu\nu} \frac{1}{n^\mu(\mu)} \ ,$$

and so,

$$\delta_D n^\nu(\mu) = \xi(\mu)\delta_D g^{\mu\nu}\frac{1}{n^\mu(\mu)} + \xi(\mu) g^{\mu\nu}\delta_D \frac{1}{n^\mu(\mu)} \ .$$

This expression can be arranged to give

$$\delta_D n^\nu(\mu) = \delta_{\rm naive} n^\nu(\mu) - \Big(n_\mu(\mu) g^{\nu\sigma} - \xi(\mu) n_\mu(\mu) n^\nu(\mu) n^\sigma(\mu)\Big)\partial_\sigma \epsilon^\mu \ ,$$

which, using (4), is

$$\begin{aligned}\delta_D n^\nu(\mu) &= \delta_{\rm naive} n^\nu(\mu) - n_\mu(\mu)\gamma^{\nu\sigma}(\mu)\partial_\sigma\epsilon^\mu \\ &= \delta_{\rm naive} n^\nu(\mu) - n_\mu(\mu)\delta^\nu_a\gamma^{ab}(\mu)\partial_b\epsilon^\mu \ ,\end{aligned} \tag{65}$$

for $a, b = 0, 1, 2, 3$ *except* μ. Equation (65) expresses the deviation of the transformation of $\mathbf{n}(\mu)$ from the "naive" vector behavior. Notice that this deviation differs from zero only for infinitesimal diffeomorphisms ϵ^μ such that $\partial_b\epsilon^\mu \neq 0$; these are the diffeomorphisms that do not preserve the foliation of ${}^4V$ in surfaces $x^\mu = constant$.

7.2 Proof of (14)

Let us single out, for instance, the $\partial_0\partial_3$ terms in

$$\mathcal{L}_K = \sqrt{|g|}R + 2\partial_\mu(\sqrt{|\gamma(\mu)|}K(\mu)) - \partial_\mu\partial_\nu(\sqrt{|\gamma(\mu\nu)|}\alpha(\mu\nu)) \ .$$

To this end, it is convenient to use the ADM decomposition

$$\sqrt{|g|}R = \mathcal{L}_{ADM} - 2\partial_\mu\left(\sqrt{|g|}(n^\lambda(0)n^\mu_{;\lambda}(0) - n^\mu(0)n^\lambda_{;\lambda}(0))\right)$$

where

$$\mathcal{L}_{ADM} := -\sqrt{|\gamma(0)|}n_0(0)({}^3R + K_{ab}(0)K^{ab}(0) \ - K^2(0))$$

is free from $\partial_0\partial_3$ terms. ${}^3R$ is the scalar curvature for the surface $x^0 = constant$.

We will use the notation $[something]_{|_\mu}$ to isolate the additive terms in $[something]$ that contain a ∂_μ derivative, and similarly for $[something]_{|_{\mu\nu}}$. So

$$\begin{aligned}[\mathcal{L}_K]_{|_{03}} &= -2\partial_0\left(\sqrt{|g|}[n^\lambda(0)n^0_{;\lambda}(0) - n^0(0)n^\lambda_{;\lambda}(0)]_{|_3}\right) \\ &- 2\partial_3\left(\sqrt{|g|}[n^\lambda(0)n^3_{;\lambda}(0) - n^3(0)n^\lambda_{;\lambda}(0)]_{|_0}\right) \\ &+ 2\partial_0\left([\sqrt{|\gamma(0)|}K(0)]_{|_3}\right) + 2\partial_3\left([\sqrt{|\gamma(3)|}K(3)]_{|_0}\right) - 2\partial_0\partial_3(\sqrt{|\gamma(03)|}\alpha(03)) \ .\end{aligned} \tag{66}$$

The first and third term in the right side cancel because $n^0_{;\lambda}(0) = 0$ and $n^\lambda_{;\lambda}(0) = -K(0)$. On the other hand, the piece in the second term $n^\lambda(0)n^3_{;\lambda}(0)$ can be expressed as $n^\lambda(0)n^3_{;\lambda}(0) = n_0(0)\gamma^{3j}\partial_j n^0(0)$, that has no ∂_0 derivative. Therefore

$$[\mathcal{L}_K]_{|_{03}} = 2\partial_3\left([n^3(0)n^\lambda_{;\lambda}(0) + \sqrt{|\gamma(3)|}K(3) - \partial_0(\sqrt{|\gamma(03)|}\alpha(03))]_{|_0}\right) \ , \tag{67}$$

Now, a little algebra gives

$$\begin{aligned}[n^3(0)n^\lambda_{;\lambda}(0) + \sqrt{|\gamma(3)|}K(3)]_{|_0} &= \sqrt{|\gamma(3)|}n^0(3)(ln\frac{n^0(3)}{n^0(0)})_{,0} \\ &= \sqrt{|\gamma(03)|}\frac{1}{\sqrt{1+q^2(03)}}\partial_0 q(03) \\ &= \sqrt{|\gamma(03)|}\partial_0\alpha(03) \ ,\end{aligned} \tag{68}$$

and, finally, plugging this result in (67),

$$[\mathcal{L}_K]_{|03} = -2\alpha(03)\partial_0\partial_3(\sqrt{|\gamma(03)|}) \ . \tag{69}$$

The ADM decomposition gives also an immediate proof that $\partial_0\partial_0$ terms are not present in $\mathcal{L}_K$. An extension [13] of the ADM decomposition —which is associated with the $\mu = 0$ coordinate— to any other coordinate helps to prove, along the same lines, that

$$[\mathcal{L}_K]_{|\mu\mu} = 0 \ , \tag{70}$$

and that

$$[\mathcal{L}_K]_{(\text{second order terms})} = -\alpha(\mu\nu)\partial_\mu\partial_\nu(\sqrt{|\gamma(\mu\nu)|}) \ . \tag{71}$$

7.3 Proof of (51)

Recalling (8), for δ being the time derivative,

$$\tilde{g}^{\mu\nu 0}_\sigma \dot{\mathbf{\Gamma}}^\sigma_{\mu\nu} = -g_{ij}\dot{P}^{ij} + \partial_i(\sqrt{|g|}g^{00}\partial_0(\frac{g^{0i}}{g^{00}})) \ , \tag{72}$$

and recalling also (10) (with δ being the time derivative)

$$\sqrt{|g|}\left(g^{00}\partial_0(\frac{g^{0i}}{g^{00}}) + g^{ii}\partial_0(\frac{g^{0i}}{g^{ii}})\right) = 2\sqrt{|\gamma(0i)|}\partial_0\alpha(0i) \ , \tag{73}$$

we can write

$$g_{ij}\dot{P}^{ij} - 2\partial_i(\sqrt{|\gamma(0i)|}\partial_0\alpha(0i)) = -\tilde{g}^{\mu\nu 0}_\sigma \dot{\mathbf{\Gamma}}^\sigma_{\mu\nu} - \partial_i(g^{ii}\partial_0(\frac{g^{0i}}{g^{ii}})) \ . \tag{74}$$

Considering [21] the following identity with $X^\mu = (1,0,0,0)$, that originates from the definition of the Riemann tensor,

$$\dot{\mathbf{\Gamma}}^\sigma_{\mu\nu} = \nabla_\mu\nabla_\nu X^\sigma - R_{\lambda\mu\nu}{}^\sigma X^\lambda \ , \tag{75}$$

then

$$\begin{aligned} \tilde{g}^{\mu\nu\rho}_\sigma \dot{\mathbf{\Gamma}}^\sigma_{\mu\nu} &= \sqrt{|g|}(\nabla_\mu\nabla^\mu X^\rho - \nabla_\mu\nabla^\rho X^\mu - g^{\mu\nu}R_{\lambda\mu\nu}{}^\rho X^\lambda + g^{\rho\nu}R_{\lambda\mu\nu}{}^\mu X^\lambda) \\ &= \partial_\mu\left(\sqrt{|g|}(\nabla^\mu X^\rho - \nabla^\rho X^\mu)\right) + 2\sqrt{|g|}R_\lambda^{\ \rho} X^\lambda \ , \end{aligned} \tag{76}$$

and, for $X^\mu = (1,0,0,0)$ and $\rho = 0$,

$$\begin{aligned} \tilde{g}^{\mu\nu 0}_\sigma \dot{\mathbf{\Gamma}}^\sigma_{\mu\nu} &= \partial_\mu\left(\sqrt{|g|}(\nabla^\mu X^0 - \nabla^0 X^\mu)\right) + 2\sqrt{|g|}R_0^{\ 0} \\ &= \partial_i\left(\sqrt{|g|}(\nabla^i X^0 - \nabla^0 X^i)\right) + 2\sqrt{|g|}R_0^{\ 0}. \end{aligned} \tag{77}$$

Then,

$$\begin{aligned} g_{ij}\dot{P}^{ij} - 2\partial_i(\sqrt{|\gamma(0i)|}\partial_0\alpha(0i)) &= -\tilde{g}^{\mu\nu 0}_\sigma \dot{\mathbf{\Gamma}}^\sigma_{\mu\nu} - \partial_i(g^{ii}\partial_0(\frac{g^{0i}}{g^{ii}})) \\ &= -2\sqrt{|g|}R_0^{\ 0} - \partial_i\left(\sqrt{|g|}((\nabla^i X^0 - \nabla^0 X^i) + g^{ii}\partial_0(\frac{g^{0i}}{g^{ii}}))\right) \\ &= -2\sqrt{|g|}R_0^{\ 0} + 2\partial_i\left(\sqrt{|g|}\gamma^{0\mu}(i)\mathbf{\Gamma}^i_{0\mu}\right). \end{aligned} \tag{78}$$

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